Problem Set 4 (Due on Monday, 11/08)

Problem 1 *Let G be a connected simple graph with $V = \{v_1, \dots, v_n\}$ and $d_i = \deg v_i$ for every $i \in [n]$. In terms of $(d_1, \dots, d_n)$, find the minimum number of edges that must be added to G to obtain a graph with an Eulerian circuit.*

Problem 2 *Let G be a simple graph on n vertices. Suppose that $\deg v + \deg w \ge n$ for each pair of distinct non-adjacent vertices v and w of G.*

1. *Prove that G is connected.*
2. *Prove that G has a Hamiltonian cycle.*

Problem 3 *For what values of n, can we decompose K_n into the union of edge-disjoint Hamiltonian cycles?*

Problem 4 *Let T be a tournament that is not strongly connected. Prove that the set of vertices of T can be partitioned into nonempty subsets A and B such that all edges between A and B go from A to B.*

Problem 5 *A plane rooted tree is a tree with a distinguished vertex, usually called **root**, with a total ordering on the children of each vertex. For each $n \in \mathbb{N}$, find the number of plane rooted trees with n edges.*

Problem 6 *Let k be a positive integer, and let T be a tree with precisely one vertex of degree j for every $j \in [\![2, k]\!]$. Find the number of vertices of T if the rest of the vertices of T are leaves (i.e., have degree one).*

Problem 7 *We say that a tree T is **trivalent** if it satisfies that $\deg v \in \{1, 3\}$ for all $v \in V(T)$. Let T be a trivalent tree with ℓ leaves.*

1. *Find the number of three-degree vertices of T.*
2. *Prove that if $\ell > 3$, then there is a vertex of T that is adjacent to two leaves.*

Problem 8 *Let G be the graph obtained from the complete graph K_n by removing an edge. Find the number of spanning trees of G.*